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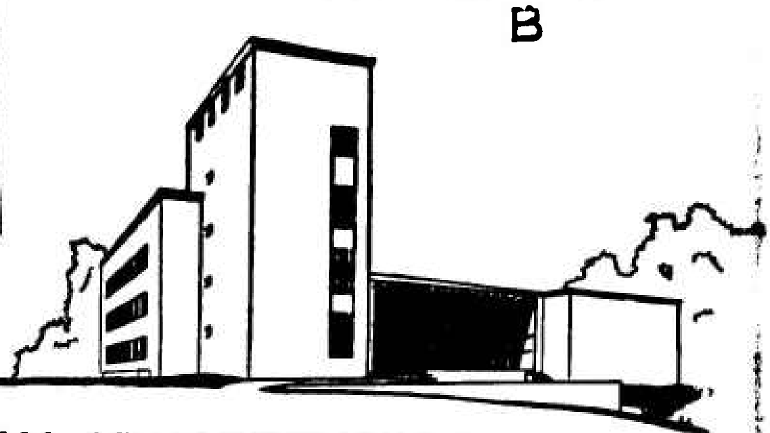

## GRADUATE SCHOOL of INDUSTRIAL ADMINISTRATION

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# SOME NETWORK CHARACTERIZATIONS FOR MATHEMATICAL PROGRAMMING AND ACCOUNTING APPROACHES TO PLANNING AND CONTROL

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## I. Introduction: [1]

In this paper we shall undertake to explore some of the ways in which accounting and mathematical programming might be related. The network type models [2] of ordinary linear programming will therefore be singled out as a natural avenue for effecting these contacts via the very elementary double-entry constructs which underlie accounting. This is to say that we will here endeavor to effect these contacts only at these very elementary levels and hence we shall restrict the discussion accordingly. The present paper is therefore best regarded as supplying only a series of sketches and suggestions for routes that might be further developed along these, or similar, lines for subsequent use in possibly improved approaches to problems in management planning and control. As will shortly be seen, this will involve, in any case, extensions, or at least altered interpretations and applications, of certain traditional ideas and uses if routes such as we are suggesting are to be followed to these ends.

As a first example we shall examine some of the ideas of PERT-Critical Path Scheduling. The latter generally proceed by reference to so-called

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1/ Numbers in brackets refer to the similarly coded references in the bibliography at the end of this paper. Note: some of these references have been supplied as a convenience for those who might wish to follow up on various aspects of the topics to be covered in this paper.

2/ See Chapter I and passim in [ 16 ] for further discussion of the idea of "model types". See also [ 19 ].

"project graphs" which will here be transformed into corresponding networks.[1/] This will then enable us to exhibit these PERT-Critical Path ideas in the form of certain linear programming problems. This route, as we shall see, offers certain advantages[2/] in that it utilizes certain "incidence relations" which also provide a convenient way to effect some of the indicated contacts with accounting.

Thus, via these contacts,which the indicated incidence relations supply, it should be possible to develop various ways in which accounting might be used in association with such optimizations as when, for instance, a critical path analysis[3/] is required by the solicitation to bid on a government contract. The way that we shall proceed is also designed, however, to indicate shortcomings in some of these PERT-Critical Path constructs when viewed from an accounting standpoint. For instance, the focus in such analyses is rather naturally directed to project tasks or components which are critical--e.g., from a time-of-delivery standpoint--and this, in turn, leads to what might be called "selective accounting."[4/]

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1/ E.g., as was done in [ 17]. See also Chapters XVII and XIX in [ 16] for discussion of related topics and approaches.

2/ Including contacts between PERT-Critical Path Analyses and linear programming so that the theory and procedures of the latter may be used to effect further extensions and to supply a route for securing improved algorithms for the former. See, e.g., [ 17].

3/ Possibly implemented by means of one of the analogue computers that are available for these purposes.

4/ These ideas appear to be related to the distinction between "causal" and "classificatory" uses of double-entry accounting as used by Ijiri in [ 55].

By this we mean that the double entry routine is applied only to these "critical aspects" of the total program whereas, as should be evident, accounting may also be required to consider related tasks and times which are <u>not</u> critical but which must nevertheless be recorded and reported for other purposes of planning and control.

To be sure, the latter requirements can also be accommodated and this can be done in numerous ways by evident extensions and adaptations of the ideas we shall introduce in association with these Critical Path ideas. Even when this is done, however, certain shortcomings will still remain and these will require attention for still further uses in accounting. Hence we shall not here examine any of the ways in which such extensions might be effected for the accounting rules and interpretations that we shall associate with analyses of the PERT-Critical Path type. Instead, we shall proceed to extend classical network constructs in order thereby to accommodate some of the more general situations that may be encountered in accounting applications.[1/]

Conversely we shall want to extend or at least alter some of the customary accounting applications of double-entry analysis in planning and control contexts. Thus, in particular, we shall want to relax the usual accounting presentations which proceed only in terms of a single dimension (e.g., dollars) and replace these with ones that proceed by reference to whatever dimensions are pertinent for the plans or controls that are to be implemented. For instance, at some point a projected program

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1/ Vide [16] Chapter XVII for further remarks and illustrations.

(or plan) may best be presented in financial terms while at other points the presentation may be better oriented toward physical units. This is all to be done analytically, however, with reference to the underlying models and the double-entry discipline which can be associated with these models via the routes that we shall indicate. For this purpose, then, it is evidently desirable to extend the use of such models from a "selective" to a more "complete" accounting basis where the latter terms is now intended to include transformations to the various dimensions that are deemed to be relevant for the indicated plan or program.

We shall also see that this "complete accounting" extends to various types of costing and evaluation schemes which are needed for a variety of managerial purposes. For instance, historical costs or their projections might then be employed for purposes of delineating the program possibilities under indicated policies and conditions. They may evidently also be needed for purposes of control as well when, for instance, a particular program has been selected from among the available alternatives. On the other hand, when conditions are to be altered so that still further alternatives can be admitted then the approaches we shall designate must also yield the opportunity costs (in physical or dollar terms) that are then pertinent and required for these considerations.

These ideas all seem to be germane for considering possible further extensions of accounting applications to problems in management planning and control. Hence, although we shall proceed by examples in subsequent sections these will all be developed so that their more general impact may be understood and assessed. Before proceeding to these examples, however, it seems best to undertake some preliminary definitions and characterizations, as is done in the next section. Also, then, after the examples are presented, we may usefully return to some of the topics covered in the present section and examine them somewhat more sharply in the light of these same examples. This should then enable us to conclude by reference to the impact that these developments might have for (a) present practices and (b) future courses of possible development in accounting.

## II. Some Terminology and Concepts:

Formally speaking, mathematical programming involves an optimization of one or more functions under one or more sets of constraints. The indicated functions each generate a "figure of merit" for which a best possible value is sought. The constraints impose conditions which reflect managerial policies as well as natural laws, technological conditions and social customs or legal restraints.

To make the sense of this more concrete we may visualize these characterizations in terms of a process of management planning which proceeds as follows. First the top management of a certain firm is required to supply policy guides for the use of staff enegaged in planning a sales campaign. For concreteness, we may suppose that these policies require the attainment of breakeven volumes 1/ in each major product line

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1/ See Ijiri [ 54 ] for further elaborations in terms of "goal programming" formulations and constructs.

after allowance for loss leaders, promotions, advertising, etc., within each such major product category.

Given these policies the planning staff is then supposed to provide some orderly arrangement for examining the alternatives that these policies admit. For instance, machine capacities, warehouse limitations. union and customer-supplier contracts, market potentials, etc., might then all be arrayed in the form of a set of "constraints" in each relevant location. Such constraints naturally include the indicated breakeven conditions. In any case, however, certain relations are thereby prescribed which are designed to show how the indicated "stipulations" 1/ can be fulfilled by reference to the variables that are incorporated in each of these relations.

Each such relation between variables and a stipulation forms a constraint. Generally speaking there will be many such constraints and also there will be many ways in which the indicated constraints may be met. Each choice of variables which satisfies the constraints forms a "program" and hence the latter has as its mathematical correspond a "solution" 2/ in the sense of a set of variable values which satisfy all of the indicated constraints.

To effect a choice among these program possibilities, the management of this firm may supply still further guides which can be formulated in

1/ This terminology is adapted from [ 16] Chapter I.

2/ These are sometimes further distinguished as "feasible solutions."

terms of so-called "figures of merit."[1/] To see what this means we shall suppose that this management is interested in market penetration. More precisely, its program orientation is toward the sales volumes that will be generated in each of the relevant markets. Bearing the breakeven relations and other constraints in mind the planning staff might then proceed to examine the sales volumes generated by the various "feasible" programs. In particular each of the relevant physical volumes might then be priced by reference to the sales projections that are implied by the indicated programs. These unit sales prices (including discount allowances, etc.) are referred to as "criterion elements" (or, more briefly, "criteria") which can be applied to the physical volumes in order to generate the relevant figures of merit. Thus, in this case, total dollar sales in each relevant market provides one of the indicated "figures of merit."

Of course, the latter can be used to array[2/] the various programs in accordance with the figures of merit that they generate. Alternatively, the management of this firm can proceed a step further and state an "objective" such as "choose the program which maximizes these figures of merit." Note now that this latter statement involves two parts: (1) a statement of what is wanted which is here accorded the form of a "maximization" and

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1/ This term is borrowed from the literature of operations research. See, e.g., [ 16], Chapter I.

2/ Note, however, that there may be problems in effecting such arrays when more than a single overall figure of merit is used. See, e.g., Chapter IX in [ 16].

(2) an associated set of values, called "figures of merit," which indicate how the programs are to be judged in effecting a selection between the programs.

The methods of mathematical programming admit still wider ranges of possibilities, of course, and, as already indicated, some care must be used to avoid spurious choices--e.g., between historical and opportunity costs[1/]--which no longer need to be made under the methodologies that are now available. On the other hand, these possibilities ought to be balanced against the information supplying potential--in both speed and quantity--which are also available via the modern computer-information processing facilities and which are often utilized in association with such models. The advent of "total-information-instantaneous display"[2/] arrangements will, of course, make these balance issues even more moot but we shall hereafter suppose that procedures can be devised for aiding any management so that, over time, an accumulated experience will supply the assistance required to achieve the requisite balance and assistance in reaching a statement of management objectives.

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1/ Reference to the preceding section will indicate that these methodologies suggest that this might better be rephrased in terms of how the 2 might both be utilized to best advantage. (Such alterations in issues must, of course, be generally expected when applications of new methodologies are involved.)

2/ E.g., via the on-line real-time computers, display and time-sharing facilities that are now becoming available.

Supposing that these problems have all been considered, however, we may now proceed to formalize what has just been said by also supposing that the indicated figures of merit can all be obtained from a series of specifically formulated functions--viz., $f_1(x_1,\ldots,x_n)$, $f_2(x_1,\ldots,x_n),\ldots$, $f_k(x_1,\ldots,x_n)$. Here, in the present illustration, $f_1(x_1,\ldots,x_n)$ may be supposed to refer to total dollar sales in market 1 while $f_2(x_1,\ldots,x_n)$ refers to total dollar sales in market 2, and so on, for each of k markets that are of interest. That is, these functions transform any choice of $x_1,\ldots,x_n$ values into a corresponding sales value for each of the relevant markets. Of course, these $x_1,\ldots,x_n$ choices must conform to all constraints that are imposed. Provisionally we shall write three as $g_i(x_1,\ldots,x_n) \le b_i$ $i=1,\ldots,m$, so that the stipulations $b_i$ are displayed explicitly for each of these m constraints. Thus, some of the $g_i$ functions generate values which are related to the breakeven policies by, say, the lump-sum expenditures, $b_i$ which are stipulated as necessary in each relevant market. Other $g_i$ functions may be related to technological conditions and, of course, still other limitations to the choices $x_1,\ldots,x_n$ may be reflected in the constraints.

Proceeding now to a more general formulation we may write

$$\begin{array}{llc} & \text{optimize} & \{f_1(x_1,\ldots,x_n),\ \ldots\ f_k(x_1,\ldots,x_n)\} \\ & \text{subject to} & \\ (1) & & g_i\ (x_1,\ldots,x_n) \ge 0 \\ & & i=1,\ \ldots,\ m\ . \end{array}$$

as the class of mathematical programming problems that are of interest. Here the $b_i$ stipulations are incorporated in the $g_i$ functions from which the constraints are formed. This may be done without loss of generality. Also, in this same spirit of generality, we have stated the general objective as an optimization relative to a collection of k functions,

$f_j$--$j=1,2,\ldots,k$--each of which defines a corresponding figure of merit for the indicated optimization. I.e., mathematically speaking we are here optimizing a vector which has the indicated functions as its components.[1/]

The model stated in (1) has been formulated so that it can accommodate situations (e.g., in governmental management)[2/] where the figures of merit are not commensurable--hence cannot be compared dimensionally. Such cases--which are said to involve multi-dimensional objectives[3/]--should not be confused with the ones that we shall employ in the examples that follow. To be sure, these examples will proceed to illustrate different dimensions in which pertinent aspects of a plan might be stated. They are nevertheless all oriented with respect to a scalar (or unitary) objective which is defined relative to a single $f(x_1,\ldots,x_n)$. They are also further restricted to the simple linear case--viz., $f(x_1,\ldots,x_n) = \sum_{j=1}^{n} c_j x_j$ --and the constraints are also characterized by reference to linear inequalities or equations--viz., $g_i(x_1,\ldots,x_n) = \sum_{j=1}^{n} a_{ij} x_j, \leq b_i$ all $i$. That is, the examples that follow all replace the general situation (1) with the following special one:

$$\text{maximize } z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$(2) \qquad b_i \geq \sum_{j=1}^{n} a_{ij} x_j$$

$$x_j \geq 0 .$$

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1/ Vide, e.g., the discussion of "functional efficiency" in Chapter IX of [ 16].

2/ See, e.g., the survey of chance constrained programming approaches to cost-effectiveness that is supplied in [ 25].

3/ These might also be characterized as "multiple objectives".

This is a linear programming problem with objective "maximize z" 1/ under the indicated constraints. There are a variety of reasons for restricting our attention to this simpler class of models. For instance, this will make it possible to obtain ready access to the duality theory of linear programming which proceeds by reference to the same data (all of it) and alters the above problem to the following associated dual

$$\text{minimize} \quad v = \sum_{i=1}^{m} w_i b_i$$

subject to

(3) $$c_j \leq \sum_{i=1}^{m} w_i a_{ij}$$

$$w_i \geq 0 \text{ .}$$

Notice now that a new figure of merit, v (for "value"), emerges and that these are related to new variables $w_i$ (for "worth of i") which are assigned to the stipulation constants $b_i$ of the original problem (2). The latter now appear as coefficients in the function which defines v. Also these new variables, $w_i$, are oriented relative to the criterion constants, $c_j$, obtained from the function in (2) and related to the constants $a_{ij}$--also from the original problem--in a form which does not permit $w_i$ choices that will cause these expressions to fall below any of the criterion constants $c_j$ which management has previously prescribed for use in the indicated plans.

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1/ I.e., z is the figure of merit.

These $w_i$ values supply the so-called "dual evaluators" which can be used for opportunity cost evaluations in whatever dimensions are pertinent relative to the programs obtained from (2). The latter--i.e., the $x_j$ values--must, of course, conform to the constraints imposed by (2) when the optimum figure of merit, z, is attained. Moreover, the methods of solution generally arrange to supply solutions to <u>both</u> problems when <u>either</u> one is solved. This means that program values, $x_j$, as well as evaluators, $w_i$, will be available for planning under stipulated conditions, on the one hand, and for evaluating the consequences of altering these conditions on the other hand. Note, for example, that the criterion elements, $c_j$, in (2) are accorded the status of stipulation constants in (3). Thus, these explorations and evaluations may be readily extended to these $c_j$ values as well as the $b_i$ constants which form the stipulations in (2).

In order to conclude these prefatory remarks we may now turn from "planning"--e.g., the problems of selecting and evaluating alternatives--in order to focus on the way in which we shall approach the topic of "control." First we should observe that the activities that are subsequently engendered via a plan will, in general, depart from almost <u>any</u> program that may be selected. The actual operations (i.e., the decisions which are made when resources are actually committed) will then give rise to variances from the designated plan. This is by way of saying that the problems of securing "conformance" between plans and operations form the core of the problem of control. This, in any event, is the way we shall approach the topic of control. We shall not here develop this topic in detail,[1/] however, but shall, instead, confine ourselves to the evident observation that a plan, once adopted, forms one part of the indicated controls and hence it is well to consider the pertinent (possibly different) dimensions which may be accorded to the plan in the records and reports that will subsequently be used relative to the decisions that are supposed to be made at each of a variety of different stages in its execution.

## III. Programming and Accounting for PERT and Critical Path Analysis:

The topic of interest for this section is now introduced via the diagram of Figure 1.

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1/ This would require recourse to "motivational costs" and the related behavioral sciences ideas as developed by A. Stedry [ 84 ] and N. C. Churchill [ 29 ].

This diagram becomes a "project graph" of the kind used in PERT-Critical Path analyses when it is viewed in the following way. The nodes represent "states" in a certain project which can be reached via the indicated arrows. The latter are associated with tasks which must be carried out in the indicated order to reach these nodes. The times required for these tasks are positioned alongside the arrowheads--e.g., the task which moves the project from the initial state (node 0) to the state at node 2 requires 20 minutes and the task that moves the project from node 2 to node 3 requires 30 minutes.

It is important to observe that the task emanating from any particular node cannot be initiated until all prior tasks leading into this node have been completed. For instance the task that moves the project from node 3 to node 6 cannot be initiated until node 3 has first been reached via nodes 0 to 2 to 3 and nodes 0 to 1 to 3. The latter requires 40 (= 10 + 30) minutes and the former requires 50 (= 20 + 30) minutes. See Figure 1. Hence 50 minutes from initiation of the project at node 0 is the earliest time[1/] at which the task going from node 3 to node 6 can be started.

Node 6 represents the state of project completion and by an evident extension of what has just been said the earliest time for this is 100 minutes. This is the time obtained from the links on the critical path

1/ For this reason this is sometimes called the "early start time." See, e.g., [ 64 ].

that are indicated on the right. That is, each link which has a solid dot alongside it forms part of a chain of such links, each of which is critical. This is called the critical path (from Node 0 to Node 6) because (a) it reaches the state of project completion and (b) any alteration of the times on any of these links will affect the time in which the project can be completed.

By way of contrast, refer to some other chain such as the one going via nodes 0 to 1 to 3 to 6, which totals to 60 (= 10 + 30 + 20) minutes. An alteration of the time on the last indicated link, say, from 20 to 40 minutes will not affect the project completion time. Indeed, as has already been observed, the early start time for the task going from Node 3 to 6 is 50 minutes. Adding the 20 minutes required to traverse this link still produces a total of only 70 minutes so that an additional 30 minutes of delay on this link would still not affect the time required to complete the project--viz., 100 minutes.

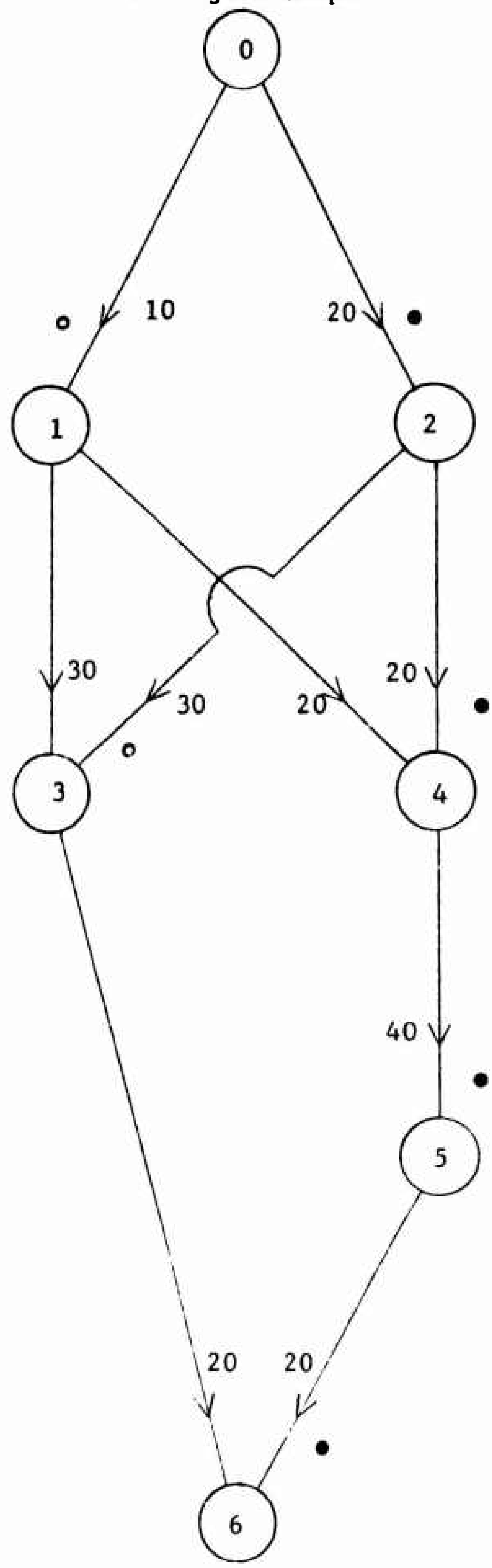
This idle capacity time of 30 minutes is referred to as "slack" and there are evidently other links (i.e., tasks) which have such slack. For instance, the link going from node 1 to node 3 has 10 minutes of slack which if added to this 30 minutes produces a total of 40 minutes of slack on the chain going from node 0 to 6 via nodes 1 and 3.

In one sense this kind of analysis can be viewed as an extension of the classical principle of exceptions. Thus, in contrast to some of the supposedly customary uses of standard costs and budgets, say, attention is here filtered and directed only to those activities which are likely to be critical. A control is thereby provided by means of which attention is directed to tasks which are critical in order to ensure conformance with an estimated (minimum) completion time for the entire project.

The results of such an analysis can evidently also be used in a variety of ways for purposes of planning as well as control. This applies not only to such problems as bidding (on projects) and scheduling but also to dimensions of cost and budgetary planning as well. For insta by means of a relatively straightforward extension of these ideas it is possible to determine (i.e., plan) how best to allocate a given amount of funds between the various tasks when the objective is to secure the greatest reduction in the time required to complete such a project.

FIGURE 1*

A Project Graph




*Source: Adapted from [17].

Instead of pursuing topics such as these, however, we merely note the possibility of incorporating them (e.g., for budgetary planning)[1/] in the types of analysis that we shall now pursue. This will then allow us to move on to the main topic of interest--viz., accounting-mathematical programming relations--and en route to this objective we now show how these PERT-critical path constructs are thereby also accommodated.

We therefore next refer to Table 1, a so-called incidence matrix formed by reference to the above graph as follows: The node and link designations are transferred from the graph and listed in any convenient order such as the one shown in Table 1. Each link, as can be observed (see Figure 1), is incident on (i.e., touches) exactly two nodes. This is reflected in Table 1 as follows. Consider any column such as, say, the first one which corresponds to the arrow of Figure 1 that is incident on nodes 0 and 1. In the row associated with the node on which the tail of this arrow is incident (i.e., node 0) a value of -1 is entered. In the row associated with the node on which the head of this arrow is incident (i.e., node 1) a value of +1 is entered. All other entries in this column receive the entry "0" (here represented as blanks) to reflect the fact that this arrow is not incident on any of the nodes shown in these rows. These same ideas are next used for the incidence relations for the link connecting nodes 1 and 3, and so on, until all links on the graph of Figure 1 have been accounted for in this manner.

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1/ See, e.g., [ 45 ] and [ 59 ].

Table 1*

Incidence Matrix for Figure 1

| Variables $w_i$ | Variables, $x_{ij}$ / Links L / Nodes, N | $x_{01}$ | $x_{13}$ | $x_{36}$ | $x_{14}$ | $x_{02}$ | $x_{24}$ | $x_{45}$ | $x_{56}$ | $x_{23}$ | Stipulations |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 0-1 | 1-3 | 3-6 | 1-4 | 0-2 | 2-4 | 4-5 | 5-6 | 2-3 | |
| $w_0$ | 0 | -1 | | | | -1 | | | | | $y_0$ |
| $w_1$ | 1 | 1 | -1 | | -1 | | | | | | $y_1$ |
| $w_2$ | 2 | | | | | 1 | -1 | | | -1 | $y_2$ |
| $w_3$ | 3 | | 1 | -1 | | | | | | 1 | $y_3$ |
| $w_4$ | 4 | | | | 1 | | 1 | -1 | | | $y_4$ |
| $w_5$ | 5 | | | | | | | 1 | -1 | | $y_5$ |
| $w_6$ | 6 | | | 1 | | | | | 1 | | $y_6$ |
| Times, $t_{ij}$ | | 10 | 30 | 20 | 20 | 20 | 20 | 40 | 20 | 30 | Total Time T |
| | | ○ | | | | ● | ● | ● | ● | ○ | |

*Source: Adapted from [ 17].

To convert this into a network with its associated analytic characterization we now adjoin the stipulations shown on the right of Table 1. The $y_i$ shown there are then assigned values that are designed to reflect the kinds of flows that are to be imposed on this network.[1] Thus, in particular,[2] setting $y_0 = -1$, $y_6 = 1$ and all other $y_i = 0$ and then dropping the $x_{ij}$ into position alongside their already assigned incidence numbers we can then regard the latter as the coefficients of these variables in the following equations:

$$
(4)\qquad
\begin{array}{rrrrrrrrrrr}
-x_{01} & & & & -x_{02} & & & & & & = -1 \\
x_{01} & -x_{13} & & -x_{14} & & & & & & & = 0 \\
& & & & +x_{02} & -x_{24} & & & & -x_{23} & = 0 \\
& x_{13} & -x_{36} & & & & & & & +x_{23} & = 0 \\
& & & +x_{14} & & +x_{24} & -x_{45} & & & & = 0 \\
& & & & & & +x_{45} & -x_{56} & & & = 0 \\
& & +x_{36} & & & & & +x_{56} & & & = 1
\end{array}
$$

Finally, the values for these variables are restricted to $x_{ij} \geq 0$ in order to ensure that the flows will all be in the directions indicated by the arrows.

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1/ Cf. Chapters XVII and XX of [ 16 ] for illustrations. We have here slightly generalized the illustration of [ 17 ] in order to accommodate some of the further accounting possibilities that may be wanted on occasion.

2/ These choices impose a unit input at node 0 and a unit output at node 6 which the flow over the network is then required to accommodate--e.g., via the associated algebraic relations as in (4).

In the bottom row of Table 1 we have also recorded the times on each link so that we may also position the $x_{ij}$ alongside these constants in order to form

$$(5) \qquad T = 10x_{01} + 30x_{13} + 20x_{36} + 20x_{14} + 20x_{02} + 20x_{24} + 40x_{45} + + 20x_{56} + 30x_{23} ,$$

an expression which provides the "figure of merit," T, associated with any solution for the system of equations (4). Then if the objective is stated as "maximize T" under these constraints we have obtained a linear programming model that can be used to obtain the maximum time for any chain of links extending from node 0 to node 6 and which will also designate the links in this chain by assigning them values of $x_{ij} = 1$ while assigning all other links the values $x_{ij} = 0$. Thus, in this case, the solution to this maximizing problem is

$$(6) \qquad x_{02} = x_{24} = x_{45} = x_{56} = 1$$

and all other $x_{ij} = 0$. See the black dots below the bottom row of Table 1. Thus the links associated with the $x_{ij} = 1$, viz., $L_{02}$, $L_{24}$, $L_{45}$ and $L_{56}$, form the chain which is critical.

Before moving to the proposed accounting characterizations it is useful to record the dual of the above linear programming. Recall that this is done by reference to new variables[1/] which are assigned to all of

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1/ The fact that neither problem involves variables from the other is an unusual feature of the duality relations in linear programming. These features provide a sharpness and convenience that is not usually available from other duality relations in mathematics.

the original constants. We have therefore positioned these new variables, $w_i$, alongside (on the left) in Table 1 so that they can also be conveniently used to form this minimization problem. That is we assign these $w_i$ to each column of Table 1 so that we may employ them in a manner that is wholly analogous to the way the previous problem was formed by reference to the rows of this incidence matrix. Thus proceeding in the indicated manner (column by column) with the same choice of $y_o = -1$, $y_6 = 1$ and all other $y_i = 0$, we obtain the following problem:1/

$$\text{minimize } g = -w_0 + w_6$$

with

$$
\begin{array}{lrcl}
 & -w_0 + w_1 & \geq & 10 \\
 & -w_1 + w_3 & \geq & 30 \\
 & -w_3 + w_6 & \geq & 20 \\
 & -w_1 + w_4 & \geq & 20 \\
(7) & -w_0 + w_2 & \geq & 20 \\
 & -w_2 + w_4 & \geq & 20 \\
 & -w_4 + w_5 & \geq & 40 \\
 & -w_5 + w_6 & \geq & 20 \\
 & -w_2 + w_3 & \geq & 30
\end{array}
$$

1/ Note that these w variables are not constrained to be non-negative. A convenient nmemonic for producing these problems in a way that takes account of alterations from equations to inequalities, etc., may be found on pp. 191-194 of [ 16 ].

This dual has a variety of uses[1/], as we shall see, but, for the moment, we need only observe that the objective, which is to minimize g--i.e., the difference ($w_6$ - $w_0$)--corresponds to minimizing the elapsed time between nodes 0 and 6 and thus produces a different (but perhaps more natural) way of looking at the problem of critical path determination. We can then record the following as a property of the duality theory of linear programming

$$\max. \quad T = \min. \ g \tag{8}$$

and confirm that this is so in the present case by recording the optimizing solution to this dual as

$$\begin{aligned} w_0 &= 0 \\ w_1 &= 10 \\ w_2 &= 20 \\ w_3 &= 50 \\ w_4 &= 40 \\ w_5 &= 80 \\ w_6 &= 100 \end{aligned} \tag{9}$$

so that min. g = $w_6$ - $w_0$ = 100 = max T, as predicted.[2/]

---

1/ Including the use made in [ 17 ], which, inter alia, showed how it could be employed to develop a highly efficient algorithm for use in critical path analysis.

2/ It is perhaps worthwhile to note, also, that (a) no $w_j$ is negative and (b) only the 2d, 3d and 4th expressions in (7) are not satisfied as equations.

The flow charts that are sometimes used in process costing,[1/] for example, bear certain resemblances to the chart of Figure 1 and this suggests a possible route for establishing such connections in this case, too. This can readily be done in a fairly simple and natural way as we shall now see after first noting that the route we have followed for this "process flow" connection yields entries which can be referred to the network and hence also to the above linear programming characterizations in a variety of ways. Thus, referring to Table 1 we may also regard it as an accounting vehicle--e.g., for use in critical path analyses--in accordance with the following conventions:

(i) Assign an account to each node

(ii) If $x_{ij} = 1$ credit and debit the amount $t_{ij}$ to the accounts with non-zero incidence numbers as follows:

(a) Credit the amount $t_{ij}$ to the account in the row which has incidence number equal to -1.

(b) Debit the amount $t_{ij}$ to the account in the row which has incidence number equal to +1.

(iii) If $x_{ij} = 0$ make no entry.

We might now illustrate the use of these rules by referring to the link $L_{02}$ which is on the critical path (see the black dot under this column in Table 1 or refer to (6 )) and hence has its associated variable $x_{02} = 1$. We can therefore draw upon Table 1 and write

---

1/ Vide e.g., [ 2 ] and [ 49 ]. For other illustrations in connection with flow chart approaches to double-entry accounting see [ 4 ]. See also [36].

$$(10) \qquad x_{02} L_{02} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hdashline 20 \end{bmatrix}$$

which, by applying the above rule, produces the accounting entry for this time value (below the broken line) of $t_{02}$ = 20 minutes which, in journal entry form, is

| | | | |
|---|---|---|---|
| Dr. Node 2 | | 20 minutes | |
| Cr. | Node 0 | | 20 minutes |

Similarly $x_{24} = 1$ produces

| | | | |
|---|---|---|---|
| Dr. Node 4 | | 20 minutes | |
| Cr. | Node 2 | | 20 minutes |

We can also apply the ordinary rules of algebraic addition[1/] to combine these two entries as $x_{02} L_{02} + x_{24} L_{24}$ to produce

$$(11) \qquad \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hdashline 20 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ \hdashline 20 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \hdashline 40 \end{bmatrix}$$

---

1/ *Vide*, e.g., Chapter V in [ 16 ].

Notice that the result of this addition also produces a column with only 2 non-zero entries at values $\pm$ 1. Hence the above rules may be unambiguously applied to produce the following entry

| | | | |
|---|---|---|---|
| Dr. | Node 4 | 40 minutes | |
| | Cr. Node 0 | | 40 minutes |

In short this addition produces the cumulative amounts for "work in process" at the state corresponding to Node 4 with the corresponding cumulative liability (in minutes) which is assigned to Node 0. Indeed, continuing in this fashion, e.g., by reference to (6), above, we would finally secure $x_{02}L_{02} + x_{24}L_{24} + x_{45}L_{45} + x_{56}L_{65}$ as

$$(12)\quad \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ --- \\ 20 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ --- \\ 20 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ --- \\ 40 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ --- \\ 20 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ --- \\ 100 \end{bmatrix}$$

which produces, it may be observed, the stipulations column of Table 1 with $T = 100$, $y_0 = -1$, $y_6 = 1$ and all other $y_i = 0$. 1/ In any event the entries $\pm$ 1 determine, as before, the assignment of the total critical path time to "liability" and "finished goods" accounts as follows:

| | | | |
|---|---|---|---|
| Dr. | Node 6 | 100 minutes | |
| | Cr. Node 0 | | 100 minutes |

---

1/ This was one reason for choosing these $y_i$ values since, inter alia, it also produces a column with non-zero values of $\pm$ 1 in only 2 places.

Evidently then it should be possible in this way to join the techniques of double-entry accounting, PERT-Critical Path analyses and mathematical programming for a variety of purposes such, as for, instance, in planning and analyzing the time requirements for a proposed project. Although the indicated value T = 100 refers to the project completion time (at a minimum) there are evidently other accounting requirements that will also need to be considered. There should generally be no trouble in relating the indicated times to a total cost of completion or, if desired, an alternate approach can be followed and the values below the broken line in Table 1 can be extended to include a variety of measures such as times, costs, etc.

These simple $\pm$ 1 conventions for effecting the indicated entries can be extended to accommodate activities that are also not on the critical path. Because this would involve a recourse to other algebraic rules[1/] for addition, multiplication, etc., we shall not explore these here. Instead we shall close this section by now referring to the dual problem (7) and its solution (9).

First we should note that this dual[2/] provides a different way of characterizing the problem of critical path scheduling--as witness, for instance, the relation (8). Second we need to remember that we are here confining the discussion only to the optimal solution.[3/] Bearing this in

---

1/ E.g., the rules of Boolean-relational algebras which are explored in Rosenblatt [ 79 ] but from the standpoint of a somewhat different set of relations to accounting. See also the signal flow graphs used in Elmaghraby [ 42 ].

2/ For some of the other types of duals that are available from network theory see, e.g., de Ghellinck [ 39 ].

3/ Other solutions are sometimes referred to as "virtual values." See, for instance, the discussion of such virtual values in Chapter XVII of [16] where they are also related to network representations of certain physical problems.

mind we can then secure certain imputations that are related to the optimum (critical) path. Moreover, by characterizing sources (or inputs) and applications (or outputs) so that they are associated with the coefficients -1 and +1, respectively, we can then also make contact with accounting practice in a way that can be used, on occasion, to augment or extend some of the more usual funds flow analyses.

Proceeding now via a chain which is not critical 1/ we may refer to the first expression in (7) which we rewrite as

$$w_1 - w_0 - 10 \geq 0 \tag{13}$$

so that the transfer of $t_{01} = 10$ to the left also conforms to the sign conventions we are using. Reference to (9) produces $w_o = 0$ and $w_1 = 10$ so that, at an optimum, the expression (13) is satisfied as an equation and can therefore be interpreted as follows: $w_1 = 10$ represents an application with source in amount $t_{01} = 10$ via the activity associated with the arrow going from node 0 to node 1. 2/ Since $w_o = 0$ it yields neither a source nor an application.

Next refer to the expression $w_3 - w_1 - 30 \geq 0$ in (7). 3/ The optimal solution values are--see (9)--$w_3 = 50$ and $w_1 = 10$ so that this inequality is fulfilled strictly. It is therefore convenient to render it in the form of an equivalent equation by adjoining a variable $s_{13}$ which is constrained to be non-negative. This is a so-called slack variable, s, which we associate with

---

1/ Here we are using the chain that can be formed from the links on the left-hand side of Figure 1.

2/ Recall that we are here dealing with physical (not financial) flows. A funds flow analysis would evidently associate a payment with this time and thereby produce an application of funds for the associated cost or expense.

3/ I.e., we are now proceeding from node 1 to node 3 in Figure 1.

the link going from 1 to 3 to produce

(14) $$w_3 - w_1 - s_{13} - 30 = 0$$

which gives $s_{13} = 10$ when $w_3 = 50$ and $w_1 = 10$. Here, then, $w_3 = 50$ represents an application secured via the 3 sources: $w_1 = 10$ via node 1; $t_{13} = 30$ via the activity associated with the arrow going from node 1 to node 3; and slack (or idle time) $s_{13} = 10$ which is imputed via the critical path.1/

Turning finally to the expression $w_6 - w_3 - 20 \geq 0$ and referring to (9) we observe that $w_6 = 100$ and $w_3 = 50$ again produces a strict inequality. Hence proceeding as before we write

(15) $$w_6 - w_3 - s_{36} - 20 = 0$$

and assign an application to node 6 in the amount $w_6 = 100$ which is accounted for via the following sources: $t_{06} = 20$, $w_3 = 50$ and $s_{36} = 30$. Evidently $s_{13} + s_{36} = 10 + 30 = 40$ minutes equals the total amount of slack on this chain from node 0 to node 6.

The linear programming solution proceeds by means of what are called "trees."2/ To see what this means refer to Figure 1 and observe the

---

1/ For accounting purposes this and other slack might be accumulated in one account--e.g., an account analogous to "waste" or "scrap allowance"--or separate accounts might be provided for each activity, as desired.

2/ More precisely these are "spanning trees" since an arrow is incident on every node of the network. See, e.g., Chapter XIX of [ 16 ] for further discussion of "trees" and their use in the theory of games--which, for present purposes, may also be characterized as a branch of mathematical programming.

hollow dots assigned to the arrows stretching from nodes 0 to 1 and 2 to 3. These branches together with those associated with the solid black dots provide the trunk and branches for such a tree.

The tree that has thus been characterized may now be used as an aid in interpreting the slack values that were secured. Thus refer to the slack value $s_{36} = 30$ and suppose it were added to the previous $t_{36} = 20$ in order to produce a new time of $\hat{t}_{36} = t_{36} + s_{36} = 50$. Given this new time for the task associated with movement from node 3 to node 6 an alternate (equally) critical path would then be available via nodes 0 to 2 to 3 to 6. If, further, $t_{13} = 30$ were altered to $\hat{t}_{13} = t_{13} +$ $+ s_{13} = 40$ then still another alternate critical path would be nodes 0 to 1 to 3 to 6. In each case, it may be observed, that the branches associated with the indicated tree provide the needed guidance by ref ence to the slack that is associated with them. To state this differently these slack values are all imputed relative to the critical path and the alternate optima that would become available by increasing the times on the lirks that follow the branches from which the slack originates.

## IV. Development of a Goods Flow-Funds Flow Network Model:

The orientation in the preceding section resulted in a "process accounting" correspond of the Critical Path-PERT problems and concepts so that the latter might be treated jointly with the former. Although the basic ideas can be generalized and extended for still other types of uses in accounting, this is not the course we shall follow. We shall, instead, proceed directly to some of the still further extensions that might be employed in accounting control and planning applications. These extensions will involve generalizations in which the non-zero incidence numbers are no longer required to be at values $\pm$ 1 only, and we shall also want to illustrate ways in which additional problems may be brought into this generalized network form by means of suitably arranged transformations and reductions. No attempt will be made to develop the corresponding transaction entries in the detail used in the preceding section because we shall now suppose that those developments may be relied upon for whatever is required to amplify the following materials for this purpose.

A simple entrance to the topics of interest may be effected via the classical "warehousing problem" of linear programming which we now formulate as [1/]

$$\text{Max. } z = -\sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{n} p_j y_j$$

with

$$(16.1)\quad \begin{array}{lrcl} n \text{ Buying Constraints:} & \sum_{j=1}^{i} x_j - \sum_{j=1}^{i} y_j & \leq & H_o - I_o \\ n \text{ Selling Constraints:} & -\sum_{j=1}^{i-1} x_j + \sum_{j=1}^{i} y_j & \leq & I_o \\ 2n \text{ Non-negativity Constraints:} & x_j, y_j & \geq & 0 \end{array}$$

---

1/ Adapted from [ 16 ] Ch. XV. For simplicity we are restricting this model to the case of a single commodity and a single warehouse. Extensions to other cases, however, may be found in [ 20 ].

Here $H_o$ is a known constant representing the capacity of the warehouse and $I_o$, also a known constant, represents the amount of initial inventory. The constants $c_j$ and $p_j$ represent, respectively, the unit acquisition cost and sales price for this particular commodity in period j. Evidently, then, z (the figure of merit) represents total net profit[1/] and "Max. z" defines the objective so that the variables $x_j$ (amount to be purchased in period j) and $y_j$ (amount to be sold in period j) are to be chosen in a way that will assign z its greatest possible value. The $x_j$ and $y_j$ values must be selected, of course, so that the constraints are all honored.

The non-negativity constraints do not appear to require any interpretation. The others may be verbalized as:

| | |
|---|---|
| Buying Constraints: | Purchases and sales must be arranged so that warehouse capacity is never exceeded |
| Selling Constraints: | Sales can only be effected from inventory on hand at the beginning of the period. |

As may now be seen, the purchase and sales arrangements are to be effected over an "horizon" of i=1, ..., n periods and, in cumulative fashion,[2/]

---

1/ More precisely it is the contribution to profit and overhead.

2/ E.g., for i=1, the selling and buying constraints are, respectively,

Selling: $y_1 \leq I_o$,

Buying: $x_1 - y_1 \leq H_o - I_o$

while for i=2, they are,

Selling: $-x_1 + y_1 + y_2 \leq I_o$,

Buying: $x_1 + x_2 - y_1 - y_2 \leq H_o - I_o$

and so on, finally, to,

Selling: $-\sum_{j=1}^{n-1} x_j + \sum_{j=1}^{n} y_j \leq I_o$

Buying: $\sum_{j=1}^{n} x_j - \sum_{j=1}^{n} y_j \leq H_o - I_o$

Vide, e.g., Chapter XV in [ 16].

the constraints are arranged to reflect in each period the conditions that have just been verbalized. That is, there are n buying constraints as well as n selling constraints yielding a total of 2n constraints--apart from non-negativity.

To bring in aspects of financial planning we now adjoin n more constraints to (16.1) in the following form:

$$(16.2) \qquad \sum_{j=1}^{i} c_j x_j - \sum_{j=1}^{i-1} p_j y_j \leq \bar{M} = M_o - \underline{M} .$$

The $p_j$, $c_j$, and the $x_j$, $y_j$ have the same meaning as before. We are assuming now that this firm does not wish to borrow.[1/] I.e., all transactions are to be financed from internal funds. Moreover cash on hand is to be maintained always at a stipulated minimal amount M. Initial cash on hand is $M_o$[2/] and $\bar{M}$ is definitional--*viz.*, it is the net amount $M_o - \underline{M}$.

Finally, (16.2) expresses the condition that in each period purchases are to be effected on a cash basis only while sales are always made on an accrual basis with the resulting receivable paid in cash only in the period subsequent to the sales realization.

Each sales transaction thus produces a receivable en route to its final payment in cash. Moreover (16)--i.e., (16.1) and (16.2)--is a linear programming problem and hence has a dual. This means that the latter may be

---

1/ Extensions to borrowing, "stretching" of payables, evaluation of receivables, etc., are treated in detail in [ 23 ]. See also [ 16 ] Ch. XV.

2/ What is involved here may be seen more clearly, perhaps, if the constraints (13.2) are also presented in the following equivalent form

$$M_o + \sum_{j=1}^{i-1} p_j y_j - \sum_{j=1}^{i} c_j x_j \geq \underline{M}.$$

used to supply evaluations of such financial (balance sheet) accounts as well as other assets (e.g., the physical inventory) and even the cost of funds that are implied by the firm's liquidity constraints (16.2). Hence extensions to the more complex dimensions of opportunity costs are available via the dual while the data utilized in (16) are all of the simplest accounting varieties.

The latter topics have been dealt with elsewhere[1/] and so we need here only note that these "dual evaluators" are also available via the network equivalent into which (16) will now be transformed. For this we first introduce new variables which are related to the physical dimensions of sales and purchase by means of the following expressions

$$\begin{aligned} g_j &= h_{j-1} - y_j \\ h_j &= g_j + x_j \ . \end{aligned} \tag{17.1}$$

Second, we also introduce new variables which are related to the dollar dimensions of sales and purchases via the expressions

$$\begin{aligned} \mathcal{F}_{j+1} &= \mathcal{R}_j + p_j y_j \\ \mathcal{R}_j &= \mathcal{F}_j - c_j x_j \end{aligned} \tag{17.2}$$

---

1/ See, e.g., [ 16 ] and [ 23 ]. Extensions to the problems of capital budgeting may be found in [ 88 ]. See also [ 11],[31],[54] and [78] for treatments relative to other financial problems.

The meaning of these new variables will be elucidated by reference to the network that we shall shortly synthesize. As a guide to this synthesis we want the incidence relations that can be obtained by substituting the above expressions in (16)--i.e. (16.1) and (16.2)--in order to produce the following linear programming problem:

$$\max \ \mathcal{T}_{n+1}$$

subject to:

$$
\begin{array}{lcccccccl}
 & & & & h_i & & & \le & H_o \\
 & & x_i + g_i & & -\,h_i & & & = & 0 \\
(18) & & -\,c_i\,x_i & & & -\,\mathcal{R}_i + \mathcal{T}_i & & = & 0 \\
 & p_i\,y_i & & & & +\,\mathcal{R}_i & -\,\mathcal{T}_{i+1} & = & 0 \\
 & -\,y_i & & -\,g_i + h_{i-1} & & & & = & 0 \\
 & & & & & x_i,\ y_i,\ g_i,\ h_i,\ \mathcal{R}_i,\ \mathcal{T}_i \ge 0 & & &
\end{array}
$$

for i=1, 2, ..., n.

The indicated substitutions have evidently produced a set of equations (from the preceding inequalities) except for the n relations $h_i \le H_o$. These latter relations, which reflect the warehouse storage limitations, simply generalize this, however, to a so-called "capacitated" network.[1/] Hence, accepting this generalization we next proceed to set

1/ See, e.g., [ 14 ] or [ 44 ].

$M_o = I_o$, and $\mathcal{F}_1 = \bar{M}$ in order to produce the pairwise relations of positive and negative incidence that are wanted for each column, and these are all obtained after

we also pair $\mathcal{F}_{n+1}$ in the function with $-\mathcal{F}_{n+1}$ in the constraints. To see how this relates to what has already been said about networks it suffices, perhaps, to note that these last steps are equivalent to regarding $\mathcal{F}_{n+1}$ as a <u>variable</u> the value of which is to be made as large as possible when it outputs from the final node distinguished by the name "sink." Also the initial inventory, $I_o$, and the net funds available for use, $\bar{M}$, as given, are then to be regarded as inflows that enter the network at separate nodes which are called "sources."

The numerical data exhibited in Table 2 below may be used to supply an illustration. Here the reference is to a 5-period model. The data at the bottom of this table provide values of $I_o = 100$ tons and $\bar{M} = M_o - \underline{M} = \$500$ which are to be assigned as inputs to the two source nodes. Thus by means of these data and by reference[1/] to (18) we synthesize the network shown in Figure 2 where the $c_i$ and $p_i$--the purchase and sales prices per ton for this good in the $i^{th}$ period--are inserted in the square boxes.

---

1/ See also (19) below.

TABLE 2*

Illustrative Data

| Period $i$ | Purchase Cost $c_i$ = \$/ton | Selling Price $p_i$ = \$/ton |
|---|---|---|
| 1 | 25 | 20 |
| 2 | 25 | 35 |
| 3 | 25 | 30 |
| 4 | 35 | 25 |
| 5 | 45 | 50 |

$H_o$ = 200 tons, warehouse storage limit

$I_o$ = 100 tons, initial inventory

$M_o$ = \$1,500, initial cash balance

$\underline{M}$ = \$1,000 minimum cash balance required

*Source: [16] p. 568

The 2 horizontal arrays of links at the top and bottom of Figure 2 will be referred to as a "goods-flow and a funds-flow[1/] axis," respectively. Note that they are measured in different units and further note that the output at the final sink, $\mathcal{F}_{n+1} = \mathcal{F}_6$, which is stated in dollars, is to be made a maximum. That is the objective is "max. $\mathcal{F}_6$." Formally, then, the problem is to find a route through the network and thereby designate

1/ Used here in the sense of cash flow. See, e.g., [3].

corresponding purchases and sales which will make this final dollar amount as large as possible.

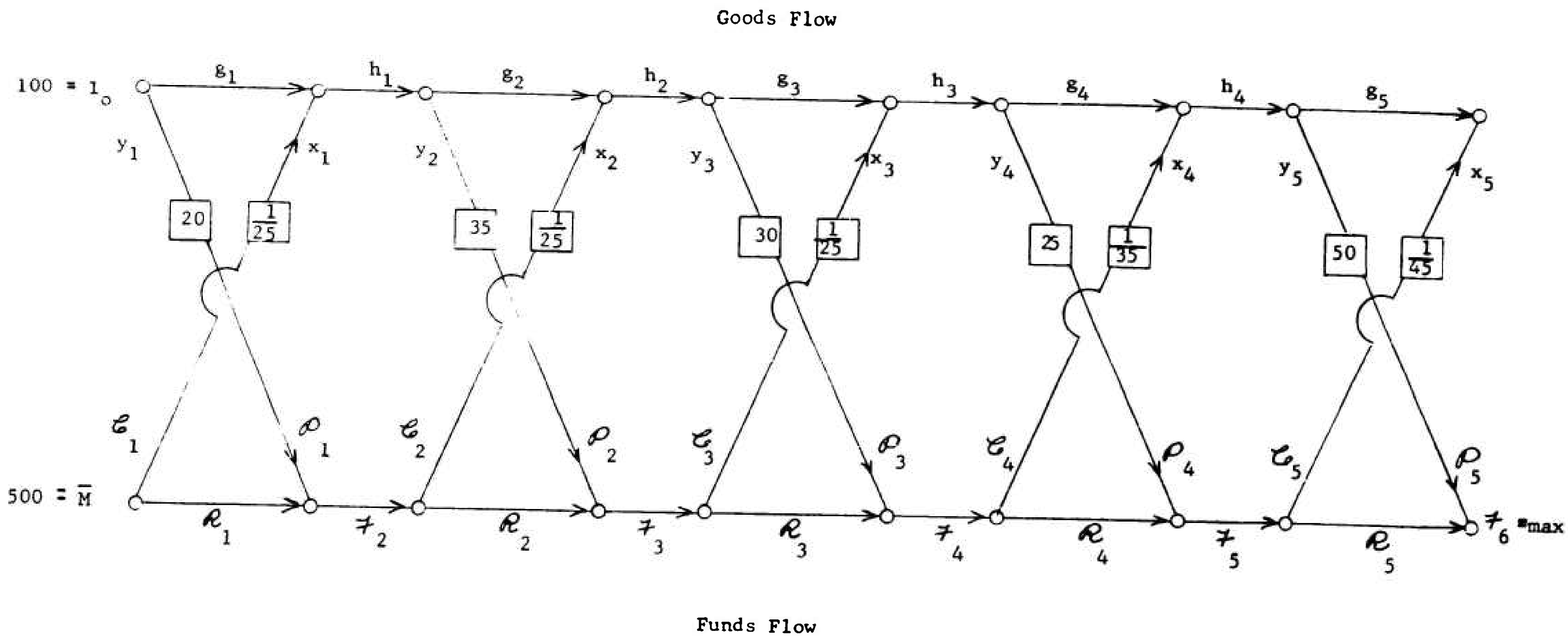


Figure 2. Warehouse-Funds-Flow Network. Note: All values $h_i \leq H_o$, the gross warehouse capacity.

Source: [16 ], p. 632.

To see what all this means we define new variables

$$
(19) \qquad \begin{aligned} \mathcal{P}_i &= p_i \, y_i \\ \mathcal{C}_i &= c_i \, x_i \end{aligned}
$$

and proceed as follows. Suppose that the entire initial inventory is sold in period 1. Then $y_1 = I_o$. Starting at the source node on the goods-flow axis, movement is then effected across the link associated with $y_1$ that leads from the goods flow to the funds flow axis. En route to the latter axis, however, a square box is encountered with the numerical value $p_1 = \$20$ exhibited there. In crossing this box the physical amount $y_1 = 100$ tons is multiplied by this unit sales price to produce a value of $\mathcal{P}_1 = \$2,000$. The latter is then added to the remant funds,[1/] $\mathcal{R}_1$, to produce the opening balance that is available for effecting purchases, if any, in period 2. I.e., this gives $\mathcal{F}_2 = \mathcal{R}_1 + \mathcal{P}_1$, all stated in dollars. See $\mathcal{F}_2$ as it relates to $\mathcal{R}_1$ and $\mathcal{P}_1$ in Figure 2.

Turning now to an analogous transition from the funds-flow to the goods-flow axis, let it next be supposed that the initial balance of $\bar{M} = \$500$ is to be used for the purchase of goods. This implies that the amount $\mathcal{C}_1 = \$500$ is to flow toward the goods axis, and, en route to there, it crosses the box with the reciprocal of $c_1 = \$25$/ton in it. An application

---

1/ Note that $\bar{M} = \$500$ denotes the net free blance initially available for effecting purchases after allowing for the minimum liquidity requirement that must be maintained--viz., $\underline{M} = \$1,000$--and hence only the increments and decrements to this balance are being considered.

of the second expression in (19) then produces $x_1 = \mathcal{C}_1/c_1 = \frac{\$500}{\$25} = 20$ tons and this amount when added to the remnant goods $g_1$ produces $x_1 + g_1 = h_1$, the number of tons of this good which is then available for sale at the start of period 2. See Figure 2.

By reference to what has just been said, the significance of the expressions in (17) as well as in (19) should also become apparent. Evidently $\mathcal{F}$ refers to funds available for effecting purchases and h refers to goods on hand from which sales can be made as can be seen from the positions of $\mathcal{F}_2$ and $h_1$ relative to the cross-over link possibilities for period 2 and the flow orientations indicated by the arrows of Figure 2. Similarly $\mathcal{R}$ and g refer to the funds and goods remaining after the transactions have been effected in any period. See, e.g., the positions of $g_1$ and $\mathcal{R}_1$ relative to the crossover possibilities for period 1 in Figure 2.

Now consider how the double entry principle would extend for use as a planning device in association with Figure 2. Thus for purposes of illustration we may refer only to the above purchase transaction and suppose that this firm is divisionalized with a corresponding set of accounts that would enable us to proceed as follows. First we transfer the requisite funds from the treasury to the purchasing division as follows:

| | | |
|---|---|---|
| Dr. Purchasing Dept. | $500 | |
| Cr. Treasury Dept. | | $500 |

Supposing that the funds are to be used for the indicated purchase we next record

Dr. Purchasing Dept. 20 tons

Cr. Purchasing Dept. $500.

Finally (assuming no sales in this period) we would transfer this to the sales department via the following entries:

| | | |
|---|---|---|
| Dr. Sales Department (to record inventory available for sale in period 2. This is $h_1$) | 120 tons | |
| Cr. Purchasing Dept. | | 20 tons |
| Cr. Closing inventory (this is $g_1$) | | 100 tons |

In order to supply each of the divisions what it needs (and no more) to govern its actions relative to the plan, we have here extended the double entry principle so that it is no longer confined to a single dimension. For this purpose note, for instance, that the sales department does not require any knowledge of acquisition costs. It only needs to know the physical quantities it will have available for sale at the start of each period. Similarly the treasurer only needs to know the amount of funds he is required to transfer, whether these are available, and the amount remaining after the transfer to purchasing is effected. All of this is supplied by the above entries. The purchasing department, on the other hand, must supply the indicated transformation from funds to goods as well as the consequences each will have for the other, and, again, this is all provided via the above entries.

Evidently other approaches can also be synthesized from the preceding relations and, of course, still other types of networks may be formed for use in different types of organizations. Here we need only observe (a) that the accounting will be "complete"[1/] whatever course is followed in the network of Figure 2 and (b) that the double-entry principle is also preserved. In elaboration of point (a) we may observe that the network of Figure 2 applies the extended incidence conventions in a way that accounts for whatever routes may be followed. In confirmation of point (b) it perhaps suffices to utilize the relations (19) in order to obtain an equivalent network stated in dollar dimensions only.[2/] Therefore substituting (19) in (18) and proceeding to some further obvious adjustments[3/] we obtain

---

1/ As contrasted, e.g., with the "selective" accountings of the preceding section.

2/ We are indebted to N. C. Churchill for suggesting the need for doing this in order to avoid an appearance of proceeding only by reference to a series of single entry recordings.

3/ It is assumed that $c_i, p_i > 0$ in all cases.

$$\max \mathcal{F}_{n+1}$$

subject to:

$$
(20)\quad
\begin{array}{lllllll}
 & & & c_i h_i & & & \leq c_i H_o \\
 & \mathcal{C}_i + c_i g_i & & -c_i h_i & & & = 0 \\
 & -\mathcal{C}_i & & & -\mathcal{R}_i + \mathcal{F}_i & & = 0 \\
\mathcal{P}_i & & & & +\mathcal{R}_i & -\mathcal{F}_{i+1} & = 0 \\
-\mathcal{P}_i & & -p_i g_i + p_i h_{i-1} & & & & = 0
\end{array}
$$

$$g_i,\ h_i,\ \mathcal{R}_i,\ \mathcal{F}_i,\ \mathcal{P}_i,\ \mathcal{C}_i \geq 0$$

This network is evidently a "financial equivalent" of the preceding model (18). Here we have used it only to show that the former, like the latter, preserves the double entry principle. Evidently either or both may be used as desired but when the latter is utilized some care may need to be exercised in considering its control consequences. Note, for instance, the evident orientation toward planning that is exhibited by the way the $c_i$ and $p_i$ values are assigned to the $g_i$ and $h_i$ (inventory) values and also to the warehouse capacity, $H_o$, in each relevant period. These variations are assigned, of course, in a manner that will exhibit the potential value variations that will accompany each of the routes that may be used en route to the final sink.

This brief discussion will perhaps serve to indicate the further possibilities that may be obtained--e.g., by extending the ordinary funds-flow analysis to accommodate both physical and financial features. There are still further possibilities available from the dual evaluators which can also be applied to both the physical and financial dimensions in ways such as will now be briefly indicated.

Consider, for instance, the opening inventory of $I_o = 100$ tons. By means of the optimal dual variables it may be ascertained that a 1-ton increment to this inventory can be made to accumulate to \$70 so that by a simple multiplication it may be found that the initial 100 tons is worth \$7000 relative to the sales and purchase opportunities that can be planned for execution. It should be noted, however, that this is the value (in dollars) that this physical inventory can be made to generate by the end of period 5. If the present value is also wanted this, too, can be obtained from the dual evaluators associated with the liquidity constraint. That is, a reference to the available dual evaluators shows that an additional dollar invested in this business can be made to accumulate to \$2.80 by the end of period 5. This result, in turn, can be readily transformed into a value V = 14/5 which corresponds to the compounded incremental 1/

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1/ I.e., this is not composed from a single average but is determined rather by reference to a series of different rates applicable in each period as determined incrementally relative to the amount to which a dollar will accumulate if it is first made available in that period. See [ 23] and [ 16] for further discussion.

"opportunity cost of money" that is implicitly present in the indicated (optimal) choices. Thus utilizing this value V = 14/5 in the form of a discount factor we obtain \$70/V = \$25 as the present worth--i.e., the value of an additional ton of inventory at the <u>start</u> of the planning period. This is to say that an additional \$25 would be equal to the present worth of an additional ton of inventory since it, too, may be made to accumulate to \$70 under the indicated alternatives.

## V. Some Concluding Remarks and Suggestions:

One route for extending some of these ideas would proceed to adjoin statistical and related risk considerations to some of the preceding characterizations. E.g., the criterion elements $t_{ij}$ which are constants in Figure 1 might be replaced with random variables and associated statistical distributions as is done in [25.a], for instance. Similarly, the criterion values $p_i$, and $c_i$, which appear in the boxes of Figure 2, might also be replaced by corresponding statistical variates for use in a variety of potential applications.1/ See, e.g., [27]. Indeed a still further extension would permit of possible ways of influencing these statistical variations--e.g., by various marketing-advertising strategies1/--and also corporate risk controls by reference to suitably formulated "chance constraints."2/ One such approach might proceed by means of the ordinary payback concept provided it is understood that this need not exclude the possibility of profit maximization. That is, for this purpose, payback is accorded the status of a constraint which is to be satisfied at a specified level of probability while profit (or, more precisely, maximum expected profit) forms the objective. See, e.g., [21] for further discussions.

Another route of extension would proceed via multi-dimensional (i.e., multiple) objectives such as were discussed in our opening section. See, for instance,

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1/ See the DEMON network models [21] which are designed for such uses in marketing new products. See also [28].

2/ See, e.g., [73], [74] and [21].

the multi-copy networks and related constructs used for designing city street networks as described in [16] [1/] where the objective is to minimize the travel times from each of <u>several</u> origins to each of several destinations. [2/]

Of course the two approaches suggested above are not mutually exclusive and one may suppose that they can also be combined in a variety of ways. Here again references may be cited to preceding work in the area of cost effectiveness which are characterized by reference to chance constrained programming for use in military applications. See, e.g., [24] and [25].

These suggestions will at least serve to indicate some of the further possibilities that can be made available from further research in both mathematical programming and accounting. Contacts obtained via routes such as we have now also suggested might then prove additionally useful for effecting direct translations of at least some of the progress that is made in either case.

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1/ See also the further references cited therein--e.g., to multi-page networks--and observe that it would not be satisfactory merely to select a design which would minimize only <u>overall</u> travel time for an entire system of city streets.

2/ It is perhaps worth observing also that these <u>optimizations</u> were originally formulated to guide and control certain <u>simulations</u>. This is by way of saying that there need be no conflict between optimization and simulation approaches of the kinds that are proposed in [5] and [70], for instance.

It is perhaps more immediately germane, however, to indicate that these same contacts might also be used in a variety of ways to extend the power and scope of accounting. At a minimum, the ideas that have already been explored should help to show how accounting might be oriented for use in connection with some of these optimization models and approaches. They may also help to suggest still further approaches such as might be secured via suitably designed computer arrangements of both analogue and digital variety.

We have also already suggested some of the limitations that might need to be considered when effecting such extensions. See, e.g., the discussion of the "selective accounting" limitations that were examined in association with the PERT and Critical Path constructs covered in section III, above. It may therefore be useful to conclude on a somewhat different note by examining some of the implications that these approaches appear to hold for selected parts of current accounting practice.

Consider, for instance, the "full cost" principle of accounting and the way it is exhibited in some of the cost-accounting allocations that are commonly made. The so-called normal burden allocations of ordinary cost accounting and flexible budgeting are a case in point. These might be characterized as obtained only by means of simplified models and methods which are restricted to representations of only "average" or "normal" situations that are rarely valid for the actual situations encountered. By way of contrast we may suggest that it is now possible to consider new approaches which will adequately reflect <u>every</u> situation (not only an average one) that might be encountered and hence arrange to effect the allocations and imputations that are really pertinent to, say, the planning alternatives that are open for consideration.

To see how this might be done we can refer to the physical and financial aspects of the model that was examined in the immediately preceding section. Thus refer to the model in (16) which was used to produce the subsequent (equivalent) ones and observe that the data utilized for its formation are all of the simplest accounting variety. No imputations or allocations are involved for such data except, of course, those that are implied by reference to any projections that might be required to reflect the future behavior of these costs and prices. Nevertheless the evaluators that were available from the associated dual problem provided what was needed in the way f opportunity costs for assessing <u>both</u> the physical and financial flows. Once again it may be observed that a dual linear programming problem contains the same data (all of it) that are present in the original (direct) problem. Any imputation or allocation must, in one way or another, refer to all available alternatives that are permitted by the relevant policies and technologies, and also any evaluation must then proceed by reference to stated criteria and objectives. Thus if these are all incorporated in the original problem then they are also all reflected in the corresponding dual.

It is perhaps now well to emphasize that a strict usage requires any opportunity cost computations to proceed by reference only to the benefits that can be secured from the <u>best</u> of the available alternatives.

Thus, even the very definition of these costs requires some recourse to a principle of optimization. Evidently, then, it is well to consider whether an explicitly formulated model might be required to provide aid and guidance for the opportunity cost computations that are desired in any particular situation. This is likely to be the case in all but the simplest managerial problems. It thus seems fair to suggest that such modelling possibilities as are required might then be exploited so that, *inter alia*, the resulting models can proceed via simple, readily available and easily understood data and still absorb the burden of providing some of the more complicated cost and benefit allocations that are also wanted. As a still further guide toward an effective modelling strategy we may suggest that it is also desirable to consider ways in which models might be employed to monitor the accuracy and completeness of the data required for their implementation [1/] and we might further suggest that these ideas are capable of extension and use for both physical and financial data as well as any other data (e.g., of a so-called qualitative variety) which might be pertinent in any of a variety of management planning and control contexts.

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1/ See, e.g., [18 ] and [19 ] for further detailed discussion of these and related ideas.

Of course, it is also important to consider ways in which accounting might be continued or extended--e.g., in order to supply (a) common data services for all of the possibly different planning models that any management might use and (b) systematic bases for tracing through, interpreting, and controlling, the further consequences of the programs that might be considered. The routes we have outlined above were selected in part, of course, for their bearing on the practice of accountancy with these customary accounting functions in mind. Notice, for instance, that the needs for opportunity cost evaluations are served by the models we are considering. Thus we would further suggest that these models might all be included as part of accounting as soon as its concepts (and procedures) are extended to include these models and related documentation as part of the system of accounts. We would personally regard this as superior to some of the alternates that have been suggested--e.g., requiring an accounting system to supply all-purpose opportunity costs, such as "the cost of capital," etc. Notice, for instance, that such models would make the latter apparent, in any event, along with the alternative planning possibilities that were considered as a basis for program selection as well as related criteria, objectives, etc.

This last point presumes, of course, that these planning models are always appropriately documented and supported so that the conditions of their formation and use can be conveniently traced. This is not always the case in current practice, of course, but this, too, can be remedied by following a course such as the one we have just outlined. Thus, in particular, the above suggestion carries with it a further implication that these models and their related documents should be submitted to the usual controls, including audit, that are customarily applied to all other parts of a firm's records. Indeed we would suppose that this kind of extension might also help to improve the planning processes--especially if the indicated audits could be carried to the point of examining and reporting on subsequent actions and their relations to the programs that were recommended along with a suitable set of findings on how the latter, in turn, relate to the models that were utilized, and so on. The evolution of management service functions and the concept of "management audit" both point in this direction.[1] Hence in these respects, too, the ideas we have outlined above appear to be compatible with a natural course of evolution for the future practice of accountancy and this, in turn, seems to be wholly consistent with ways in which other parts of the management sciences are also progressing.

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1/ Vide, e.g., [38] and [86].

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13. ABSTRACT

Network characterizations are developed for effecting contacts between accounting and mathematical programming. En route to these objectives some of the customary uses of double entry accounting are altered and related to suitable generalizations of classical network ideas such as the Kirchhoff node conservation laws. Extensions of the usual node-link incidence relations provide a basis for effecting these contacts. Concrete illustrations are supplied including a goods-flow-funds-flow model which is preceded by a simpler example involving a PERT-Critical Path application. The latter is examined in the context of a uni-dimensional physical flow involving time only while the former suggests how double entry can be extended to flows that involve a variety of different dimensions. A possibility for joint coordinated uses of programming and accounting in management planning is indicated and amplified and some of the implications for alterations in accounting practice are then examined. Suggestions for further extensions include probabilistic formulations and multi-dimensional objectives.

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